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The Quantum Fluctuations of Two Coupled Josephson Junctions with the Discreteness of Electric Charge

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Abstract The quantization of two Josephson junctions coupled via inductor with the discreteness of electric charges is proposed. The finite-difference Schrodinger equation of the circuit system has been obtained in charge representation, and the Schrodinger equation is turned to be Mathieu equation in flux representation. The wavefunction and energy spectrum can be solved by adopting the canonical transformation and unitary transformation method. The results indicate that the quantum fluctuations of the flux in the ground states of each mesh exist and are interrelated.

Keywords Coupled Josephson junction qubits · Discreteness of electron charge · Quantization · Mathieu equation · Quantum fluctuation

1 Introduction

Superconducting devices such as Cooper pair boxes, Josephson junctions, and superconducting quantum interference devices (SQUIDs) have attracted much attention in the quantum information community. Because they are relatively easy to scale up and have been demonstrated to have relatively long decoherence time [1–3], and have been considered as promising candidates for physical implementation of quantum computation [4]. Several single solid state qubits have been realized using Josephson junction circuit [5, 6]. To realize the quantum manipulation of the Josephson junction system, people have studied the energy eigenvalues, eigenstate and other quantum characteristics in detail. As a next step, coupled

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multiple qubits need to be studied [7]. The investigation indicates that all logic gates can be realized by the combination of a two-bit controlled NOT gate and a single-bit gate. Therefore, in order to fabricate the quantum logic gates that can perform arbitrary operation, the central problem are how to realize the scalable entangled coupling between two arbitrary single superconducting bits. Recently, charge quantum dynamics in coupled Josephson qubits has been observed [8]. Spectroscopic measurements on coupled phase qubits have been performed as well [9]. It is well known that the processing of quantum information is in fact a process to manipulate the quantum state, and quantum fluctuations have important influence on the manipulation of the quantum state. In this paper, based on the fundamental fact that the electric charge takes discrete values, a finite-difference Schrodinger equation of two coupled Josephson junctions was obtained. With a unitary transformation, the Schrodinger equation becomes the standard Mathieu equation. Then we study the quantum fluctuations of the flux of the junctions in the ground state.

2 Model Hamiltonian

In the case of the junction resistance neglected, two Josephson junctions coupled via inductor can be equivalent to a three-mesh mesoscopic coupled circuit as shown in Fig. 1 [10, 11]. The Hamiltonian of the system is

$$H = \sum_{j=1}^{3} \frac{1}{2L_j} p_j^2 + \frac{1}{2C_1}(q_1 - q_2)^2 + \frac{1}{2C_2}(q_3 - q_2)^2, \tag{1}$$

where L_j and C_j are inductance and capacitance of each component circuit, respectively. q_j is the charge that is regarded as a "coordinate" of the system, while their conjugation variables $p_j = L_j \dot{q}_j$ represent the flux instead of the conventional "momenta". With the electric charge being quantized as a continuous variable-"coordinate", the LC design circuit was quantized by Louisell [12]. Based on the theory, the general rules are

$$q \to \hat{q}, \qquad p \to -i\hbar \frac{\partial}{\partial \hat{q}}. \tag{2}$$

However, Li and Chen [13] pointed out that when the transport dimension reaches a characteristic dimension, i.e., the charge carrier inelastic coherence length, the discreteness of electric charge must be taken into account in mesoscopic circuit. Put in another way, the electric charge quantization may play dominant role in mesoscopic circuits, for example, the Coulomb blockade effect arises from the discreteness of electric charge. The quantization process must be reconsidered with charge discreteness. In this case, the charge operator assumes discrete eigenvalues of the form $\hat{q}|q\rangle = mq_e|q\rangle$, where m is an integer, and the elementary electric charge is $q_e = 1.602 \times 10^{-19} C.|q\rangle$ stands for the eigenstate of electric

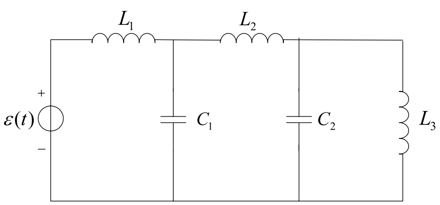

Fig. 1 Equivalent circuit of two Josephson junctions coupled via inductor

charge for circuit. From Li and Chen [13], charge discreteness is obtained from the formal change

$$q \to \hat{q}, \qquad p \to \frac{2\hbar}{q_e} \sin\left(\frac{q_e}{2\hbar} \hat{p}\right). \tag{3}$$

For the sake of convenience, we delete the superscripts of relative operators in the following discussion such as adopting q_n instead of $\hat{q}_n$. Based on the above formula, the quantized Hamiltonian of the system can be written as

$$H = \sum_{j=1}^{3} \frac{\hbar}{q_e^2 L_j} \left[\cos\left(\frac{q_e}{\hbar} p_j\right) - 1\right] + \frac{1}{2C_1}(q_1 - q_2)^2 + \frac{1}{2C_2}(q_3 - q_2)^2. \tag{4}$$

From above formula, we have obtained the following finite-difference Schrodinger equation of the circuit

$$\left\{\sum_{j=1}^{3} \frac{\hbar}{q_e^2 L_j} \left[\cos\left(\frac{q_e}{\hbar} p_j\right) - 1\right] + \frac{1}{2C_1}(q_1 - q_2)^2 + \frac{1}{2C_2}(q_3 - q_2)^2\right\} |\Psi\rangle = E|\Psi\rangle. \tag{5}$$

3 Solutions of the Finite-Difference Schrodinger Equation

Equation (5) represents harmonic oscillators that are coupled to each other. In order to solve this equation, we can introduce the following unitary operator U in the coordinate representation [14]

$$U = \frac{1}{\sqrt{\lambda_1 \lambda_2}} \iiint dq_1 dq_2 dq_3 \left| u_{jk} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \tag{6}$$

where $|q\rangle = |q_1, q_2, q_3\rangle$ is the three-mode coordinate eigenstate and

$$u_{jk} = \begin{pmatrix} 1 & -(M_1\lambda_1)^{-1}(L_2 + L_3) b \cos\alpha & (M_1\lambda_2)^{-1}(L_2 + L_3) b \sin\alpha \\ 1 & -(M_1\lambda_1)^{-1} L_3 a \sin\alpha & -(M_1\lambda_2)^{-1} L_3 a \cos\alpha \\ 1 & (M_1\lambda_1)^{-1}(L_1 + L_2) a \sin\alpha & (M_1\lambda_2)^{-1}(L_1 + L_2) a \cos\alpha \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -(M_1\lambda_1)^{-1} L_3 a \sin\alpha & -(M_1\lambda_2)^{-1} L_3 a \cos\alpha \\ 0 & (M_1\lambda_1)^{-1} L_1 b \cos\alpha & -(M_1\lambda_2)^{-1} L_1 b \sin\alpha \\ 0 & (M_1\lambda_1)^{-1} L_1 b \cos\alpha & -(M_1\lambda_2)^{-1} L_1 b \sin\alpha \end{pmatrix}, \tag{7}$$

in which

$$\lambda_1 = \left(\frac{A}{2L_2}\right)^{\frac{1}{4}}, \quad \lambda_2 = \left(\frac{B}{L_3}\right)^{\frac{1}{4}},$$
$$a = \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}}, \quad b = \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}},$$
$$M_1 = L_1 + L_2 + L_3.$$

A, B and α is determined by the following formulae respectively:

$$\frac{1}{A} = \frac{a^2\cos^2\alpha}{\mu_1} + \frac{b^2\sin^2\alpha}{\mu_2} - \frac{\sin 2\alpha}{L_2}, \tag{8}$$

$$\frac{1}{B} = \frac{a^2\sin^2\alpha}{\mu_1} + \frac{b^2\cos^2\alpha}{\mu_2} + \frac{\sin 2\alpha}{L_2}, \tag{9}$$

$$\left(\frac{b^2}{\mu_2} - \frac{a^2}{\mu_1}\right)\sin 2\alpha = \frac{2\cos 2\alpha}{L_2}, \tag{10}$$

where

$$\frac{1}{\mu_1} = \frac{1}{L_1} + \frac{1}{L_2}, \qquad \frac{1}{\mu_2} = \frac{1}{L_2} + \frac{1}{L_3}.$$

According to Fan, we can conveniently show that U is really a unitary operator; that is $UU^+ = U^+U = 1$ and $U^+ = U^{-1}$.

$$u_{jk}^{-1} = \begin{pmatrix} \frac{L_1}{M_1} & \frac{L_2}{M_1} & \frac{L_3}{M_1} \\ -\lambda_1 a\cos\alpha & \lambda_1(a\cos\alpha - b\sin\alpha) & \lambda_1 b\sin\alpha \\ \lambda_2 a\sin\alpha & -\lambda_2(a\sin\alpha + b\cos\alpha) & \lambda_2 b\cos\alpha \end{pmatrix}. \tag{11}$$

The transformations of the coordinates and momenta are

$$Q_j = \sum_{k=1}^{3} u_{jk}^{-1} q_k, \tag{12}$$

$$P_j = \sum_{k=1}^{3} \tilde{u}_{jk} p_k, \tag{13}$$

$$q_j = \sum_{k=1}^{3} u_{jk} Q_k, \tag{14}$$

$$p_j = \sum_{k=1}^{3} \tilde{u}_{jk}^{-1} P_k. \tag{15}$$

Then the Hamiltonian of the system can become

$$H' = \frac{1}{2M_1}P_1^2 + \sum_{n=2}^{3}\left\{-\frac{\hbar^2}{q_e^2 M_n}\left[\cos\left(\frac{q_e}{\hbar}\sum_{k=1}^{3}(\tilde{u})_{nk}^{-1}P_k\right) - 1\right] + \frac{1}{2}M_n\Omega_n^2 Q_n^2\right\}, \tag{16}$$

where

$$\begin{aligned} &M_2 = \sqrt{2L_2A}, \qquad M_3 = \sqrt{L_3B}, \\ &\Omega_2^2 = \frac{1}{A\sqrt{C_1C_2}}, \qquad \Omega_3^2 = \frac{1}{B\sqrt{C_1C_2}}. \end{aligned} \tag{17}$$

H' includes two Hamiltonians of independent harmonic oscillators and a free particle. Thus the term $(2M_1)^{-1}P_1^2$ can be separated off. Without question, Ω_n represents quantum frequency of the circuit, which is completely determined by the parameter of the circuit.

Using (16), the term $(2L)^{-1}P_1^2$ being separated off, the Schrodinger equation (5) is expressed as

$$\left\{-\sum_{n=2}^{3}\left[\frac{\hbar^2}{q_e^2 M_n}\left(\cos\left(\frac{q_e}{\hbar}\sum_{k=1}^{3}\tilde{u}_{nk}^{-1}P_k\right)-1\right)+\frac{\hbar^2 M_n \Omega_n^2}{2}Q_n^2\right]\right\}|\Psi(P_2,P_3)\rangle$$
$$= E(P)|\Psi(P_2,P_3)\rangle. \tag{18}$$

Now, we consider a representation in which the operators P_k $(k=2,3)$ is diagonal, called P_k presentation. The transformation of wave function between charge presentation and P_k presentation is,

$$\langle n|\Psi\rangle = \frac{q_e}{2\pi\hbar}\int_{-\pi\hbar/q_e}^{\pi\hbar/q_e} d\langle p|\Psi\rangle \exp\left(-i\frac{nq_e}{\hbar}p\right).$$

According to the relations

$$\langle p'|\frac{2}{q_e}\left[\cos\left(\frac{q_e}{\hbar}p\right)-1\right]|p\rangle = \frac{4\pi\hbar}{q_e^2}\left[\cos\left(\frac{q_e}{\hbar}p\right)-1\right]\delta(p-p'), \tag{19}$$

$$\langle p'|q^2|p\rangle = -\frac{2\pi\hbar^2}{q_e}\frac{\partial^2}{\partial p^2}\delta(p-p'). \tag{20}$$

The formula for expressing (18) in the p representation is

$$\left\{-\sum_{n=2}^{3}\left[\frac{\hbar^2}{q_e^2 M_n}\left(\cos\left(\frac{q_e}{\hbar}\sum_{k=1}^{3}\tilde{u}_{nk}^{-1}P_k\right)-1\right)+\frac{\hbar^2 M_n \Omega_n^2}{2}\frac{\partial^2}{\partial P_n^2}\right]\right\}|\tilde{\Psi}(P_2,P_3)\rangle$$
$$= \tilde{E}|\tilde{\Psi}(P_2,P_3)\rangle, \tag{21}$$

where

$$|\tilde{\Psi}(P_2,P_3,)\rangle = |\tilde{\Psi}(P_2)\rangle \otimes |\tilde{\Psi}(P_3)\rangle.$$

It is difficult to obtained the exact solution of (21). We only consider a special sample in this paper. Choosing special values of parameters in the mesoscopic circuit, so as to make

$$\tilde{u}_{nk}^{-1} = 0 \quad (n \neq k,\ n \geq 2).$$

In this condition, (21) can be divided into two equivalent Mathieu equations,

$$\left\{-\frac{\hbar^2}{q_e^2 M_2}\left(\cos\left(\frac{q_e}{\hbar}\sum_{k=1}^{N}\tilde{u}_{2k}^{-1}P_k\right)-1\right)+\frac{\hbar^2 M_2 \Omega_2^2}{2}\frac{\partial^2}{\partial P_2^2}\right\}|\tilde{\Psi}(P_2)\rangle = \tilde{E}^{(2)}|\tilde{\Psi}(P_2)\rangle, \tag{22}$$

$$\left\{-\frac{\hbar^2}{q_e^2 M_3}\left(\cos\left(\frac{q_e}{\hbar}\sum_{k=1}^{N}\tilde{u}_{3k}^{-1}P_k\right)-1\right)+\frac{\hbar^2 M_3 \Omega_3^2}{2}\frac{\partial^2}{\partial P_3^2}\right\}|\tilde{\Psi}(P_3)\rangle = \tilde{E}^{(3)}|\tilde{\Psi}(P_3)\rangle. \tag{23}$$

From (22) and (23), we can see that the two equations are similar to the LC circuit's Schrodinger equation in P_k presentation, in which the LC circuit's Schrodinger equation in the form of Mathieu function was first achieved. In terms of the conventional notations, the solutions to (22–23) are

$$\tilde{\Psi}^{+}_{l_2}(P_2) = \mathrm{ce}_{l_2}\left[\frac{\pi}{2} - \frac{q_e}{2\hbar}\tilde{u}^{-1}_{22}P_2, \lambda_2\right], \tag{24}$$

$$\tilde{\Psi}^{+}_{l_3}(P_3) = \mathrm{ce}_{l_3}\left[\frac{\pi}{2} - \frac{q_e}{2\hbar}\tilde{u}^{-1}_{33}P_3, \lambda_3\right], \tag{25}$$

or

$$\tilde{\Psi}^{-}_{l_2+1}(P_2) = \mathrm{se}_{l_2}\left[\frac{\pi}{2} - \frac{q_e}{2\hbar}\tilde{u}^{-1}_{22}P_2, \lambda_2\right], \tag{26}$$

$$\tilde{\Psi}^{-}_{l_3+1}(P_3) = \mathrm{se}_{l_3}\left[\frac{\pi}{2} - \frac{q_e}{2\hbar}\tilde{u}^{-1}_{33}P_3, \lambda_3\right], \tag{27}$$

where

$$\lambda_2 = \frac{4\hbar^2}{q_e^4 M_2^2 \Omega_2^2 \tilde{u}^{-1}_{22}}, \qquad \lambda_3 = \frac{4\hbar^2}{q_e^4 M_3^2 \Omega_3^2 \tilde{u}^{-1}_{33}}, \tag{28}$$

where the superscripts "+" and "−" denote the even and odd parity solutions, respectively, $l_2, l_3 = 0, 1, 2, \ldots$. $\mathrm{ce}(z, \lambda)$, $\mathrm{se}(z, \lambda)$ are periodic Mathieu functions. In this case, there exist infinite eigenvalues $\{a_l\}$ and $\{b_{l+1}\}$ that are not equal to zero. The energy spectrum is expressed in term of the eigenvalues a_l and b_{l+1} of Mathieu equations,

$$\tilde{E}^{(n)+}_{l_n} = \frac{\hbar^2}{q_e^2 M_n} + \frac{q_e^2}{8} M_n \Omega_n^2 \tilde{u}^{-1}_{nn} a_{l_n}(\lambda_n), \tag{29}$$

$$\tilde{E}^{(n)-}_{l_n+1} = \frac{\hbar^2}{q_e^2 M_n} + \frac{q_e^2}{8} M_n \Omega_n^2 \tilde{u}^{-1}_{nn} b_{l_n+1}(\lambda_n), \tag{30}$$

$n = 2, 3$.

It is known that the eigenvalues of the Mathieu equations are complicated, which are related to continuous fractions and trigonometric series, respectively. However, the equations can be solved explicitly by means of WKB method in the case of $\lambda_n \ll 1$ $(n = 2, 3)$. Some eigenvalue results of our calculation are given below:

$$\tilde{E}^{(n)+}_0 = \frac{\hbar^2}{q_e^2 M_n}\left[1 - \frac{2\hbar^2}{q_e^4 M_n^2 \Omega_n^2 \tilde{u}^{-1}_{nn}} + \cdots\right],$$

$$\tilde{E}^{(n)-}_1 = \frac{q_e^2 M_n \Omega_n^2 \tilde{u}^{-1}_{nn}}{8} + \frac{\hbar^2}{q_e^2 M_n}\left[\frac{1}{2} - \frac{\hbar^2}{4 q_e^4 M_n^2 \Omega_n^2 \tilde{u}^{-1}_{nn}} + \cdots\right],$$

$$\tilde{E}^{(n)+}_1 = \frac{q_e^2 M_n \Omega_n^2 \tilde{u}^{-1}_{nn}}{8} + \frac{\hbar^2}{q_e^2 M_n}\left[\frac{3}{2} - \frac{\hbar^2}{4 q_e^4 M_n^2 \Omega_n^2 \tilde{u}^{-1}_{nn}} + \cdots\right],$$

$$\tilde{E}^{(n)-}_2 = \frac{q_e^2 M_n \Omega_n^2 \tilde{u}^{-1}_{nn}}{2} + \frac{\hbar^2}{q_e^2 M_n}\left[1 - \frac{\hbar^2}{6 q_e^4 M_n^2 \Omega_n^2 \tilde{u}^{-1}_{nn}} + \cdots\right],$$

$$\tilde{E}^{(n)+}_2 = \frac{q_e^2 M_n \Omega_n^2 \tilde{u}^{-1}_{nn}}{2} + \frac{\hbar^2}{q_e^2 M_n}\left[1 + \frac{5\hbar^2}{6 q_e^4 M_n^2 \Omega_n^2 \tilde{u}^{-1}_{nn}} - \cdots\right],$$

$n = 2, 3$.

4 Quantum Fluctuation of the Flux

Under the condition $\lambda_n \ll 1$, the wave functions for the ground state of the mesoscopic circuit is

$$|\tilde{\Psi}_0(P_2, P_3)\rangle = \mathrm{ce}_0(z_2, \lambda_2) \otimes \mathrm{ce}_0(z_3, \lambda_3), \tag{31}$$

where

$$\mathrm{ce}_0(z_n, \lambda_n) = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\lambda_n \cos 2z_n + \frac{1}{16\sqrt{2}}\lambda_n^2(\cos 4z_n - 1) + \cdots.$$

So in the ground state of the system, we get the mean values and mean-square values of the P_n and P_n^2, respectively,

$$\langle P_n \rangle = 0, \tag{32}$$

$$\langle P_n^2 \rangle = \frac{\hbar^2}{2q_e^2}\left[1 - \frac{3}{2}\left(\frac{\hbar^2}{q_e^4 L_n^2 \Omega_n^2 \tilde{u}_{nn}^{-1}}\right)^2 + \cdots\right], \tag{33}$$

$n = 2, 3$. Using the contradictorily transformation of (11) and (20–21), we obtain the quantum fluctuations of flux p_n in the ground state of the system:

$$\langle p_n \rangle = 0, \tag{34}$$

$$\langle (\Delta p_n)^2 \rangle = \sum_{k=2}^{3} \frac{\hbar^2}{2q_e^2}\left[1 - \frac{3}{2}\left(\frac{\hbar^2}{q_e^4 M_n^2 \Omega_n^2 \tilde{u}_{nn}^{-1}}\right)^2 + \cdots\right]\tilde{u}_{nk}^{-2}. \tag{35}$$

From (34) and (35), we conclude that the average values of the flux in the ground state are zero, whereas the quantum fluctuations of flux exist in each mesh. Notice that, in fact, the subscript n of (34) and (35) depicts the position of per mesh in the circuit. Combining this to (14) and (15), we will easily find that the quantum fluctuations of the flux in each mesh are interrelated.

5 Conclusion

In this paper, we studied the quantization of two coupled Josephson junctions via inductor. Differing from the literature, in which it was simply treated as the quantization of coupled harmonic oscillators, taking the discreteness into account, we developed a quantum theory and obtained the finite-difference Schrodinger equation for the coupled Josephson junctions. The quantum fluctuations of flux in the system in the ground state are investigated. It is found that the quantum fluctuations of the flux in each mesh are interrelated.

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